AN OPTIMAL CONTROL FORMULATION OF PORTFOLIO SELECTION PROBLEM WITH BULLET TRANSACTION COST

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Abstract. This paper formulates a consumption and investment decision problem for an individual who has available a riskless asset paying fixed interest rate and a risky asset driven by Brownian motion price fluctuations. The individual is supposed to observe his or her current wealth only, when making transactions, that transactions incur costs, and that decisions to transact can be made at any time based on all current information. The transactions costs is fixed for every transaction, regardless of amount transacted. In addition, the investor is charged a fixed fraction of total wealth as management fee. The investor's objective is to maximize the expected utility of consumption over a given horizon. The problem faced by the investor is formulated into a stochastic discrete-continuous-time control problem.
Key words: transaction cost, continuous-discrete-time, stochastic optimal control problem

1. Introduction

Since the publication of Merton's seminal work, see Merton(1971), stochastic optimal control and stochastic calculus techniques have been widely applied to the area of finance. Robert. C. Merton iniatiated the study of financial markets using continuous-time stochastic models. Merton (1971, 1990) studied the behaviour of a single agent acting as a market price-taker who seeks to maximize expected utility of consumption. The utility function of the agent was assumed to be a power function, and the market was assumed to comprise a risk-free asset with constant rate of return and one or more stocks, each with constant mean rate of return and volatility. The only information available to the agent were current prices of the assets. There were no transaction costs. It was also assumed that the assets were divisible. In this idealized setting, Merton was able to derive a closed-form solution to the stochastic optimal control problem faced by the agent.

Since then, several authors have made contributions to the stochastic optimal control and stochastic calculus analyses of the Merton's model. Among them are Constantinides (1979, 1986), Cox and Huang (1989), Davis and Norman (1990), Duffie and Sun (1990), Dumas and Luciano (1991), Lelands (1985), Magill and Constantinides (1976).

The introduction of transaction costs to Merton's model was first accomplished by Magill and Constantinides (1976). Since then, several authors have published a number of works on Merton's model with transaction costs. To mention a few, they are Constantinides (1979, 1986), Davis and Norman (1990), Duffie and Sun (1990), Dumas and Luciano (1991), Lelands (1985). Duffie and Sun (1990) treated the proportional transaction costs with different formulation to others, which they call *discrete-continuous-time formulation.* Their formulation assumes that an investor observes current wealth when making transaction, and decisions to transact can be made at any time, but without no costs. They treated general linear transaction costs of the form $a\,W_{\tau_n} + b$, with W_{τ_n} denotes the amount of wealth transacted, and a and b are non-negatives. This paper treates Merton's model with fixed (bullet) transaction costs.

2. Formulation of the Model

2.1. **Uncertainty.** The following definitions on probability are standard.[1] It is assumed that a complete probability space $(\Omega, \mathcal{F}, P)$ is given. In addition, it is assumed that *a filtration* $\{\mathcal{F}_t : t \geq 0\}$ is also given. By a filtration is meant a family of σ- algebras $\{\mathcal{F}_t : t \geq 0\}$ which is increasing : $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$.

Definition 2.1. A filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t<\infty}, P)$ is said to satisfy the **usual hypotheses** if

1. $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$
2. the filtration $\{\mathcal{F}_t : t \geq 0\}$ is *right continuous.*

A stochastic process X on $(\Omega, \mathcal{F}, P)$ is a collection of random variables $\{X_t : t \geq 0\}$. The process X is said to be **adapted** if X_t is $\mathcal{F}_t$ measurable for each t.

Definition 2.2. A process $B = \{B_t : t \geq 0\}$ adapted to $\{\mathcal{F}_t : t \geq 0\}$ taking values in $\mathbf{R}$ is called a **one-dimensional standard Brownian motion** if

1. $B_0 = 0$, almost surely;
2. for $0 \leq s < t < \infty$, $B_t - B_s$ is independent of $\mathcal{F}_s$;
3. for $0 < s < t$, $B_t - B_s$ is $N(0, t-s)$.

[1]Some notions in this part are derived from Protter (1990)

In this study, it is assumed that the one-dimensional standard Brownian motion $B = \{B_t : t \geq 0\}$ is given on a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t < \infty}, P)$

2.2. **Security Markets.**

Definition 2.3. A riskless security is defined to be a security whose return in the future time is known with certainty. A risky security is one for which the return in the future is uncertain.

There are two securities available in the economy to an investor. One is a riskless security with fixed interest rate r, and the other is a risky security whose price is a geometric Brownian motion with expected rate of return α and rate of return variation σ^2. At time $t \geq 0$, the price processes $\{P_0(t)\}$ of the riskless security satisfy a deterministic differential equation

$$dP_0(t) = rP_0(t)dt, \tag{2.1}$$

while the price processes $\{P_1(t)\}$ of the risky security satisfy a stochastic differential equation

$$dP_1(t) = \alpha P_1(t)dt + \sigma P_1(t)dB_t. \tag{2.2}$$

There is money available for the investor in the economy as a medium of exchange and numeraire. Only money is exchangable for consumption. Money can only be acquired by selling securities, it cannot be borrowed. Let M_t denotes money holdings at time t. One unit of money can be exchanged at any time for one unit of consumption. The investor is assumed to receive no further income from noncapital sources, and starts with the initial stock of money $M_0 = 0$.

2.3. **Transaction costs.**

Definition 2.4. A portfolio transaction consists of withdrawing wealth in the form of money from the investment portfolio in the securities and then adjusting the portfolio of securities.

Trading opportunities are available continuously in time, but not without costs. Transactions costs are incurred when information is processed and a portfolio transaction is made. There are two forms of transaction costs: portfolio management fees and withdrawal costs. The investor pays a fraction $\varepsilon > 0$ of the total wealth in the securities at the beginning of each interval as a portfolio management fee. The portfolio management fee is meant to include the cost of adjusting the portfolio and the cost of processing information. For the purpose of analyses in this paper, transactions costs is the costs which incurs during withdrawing wealth from the portfolio.

Definition 2.5. A transaction costs is meant the withdrawal costs, which is a function of amount of wealth withdrawn from the portfolio.

The transaction costs function is a fixed for every transaction, regardless of amount of wealth transacted. Let Ψ be a transaction costs function. If W_{τ_n} denotes the amount of wealth withdrawn at time τ_n, then Ψ is defined by

$$\Psi(W_{\tau_n}) = \begin{cases} b > 0, & \text{if } W_{\tau_n} > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then the total transaction costs function is of the form $b+\varepsilon(X_{\tau_n} - W_{\tau_n})$, where X_{τ_n} is the total wealth at time τ_n before transaction.

2.4. **Information.** The Filtration $(\mathcal{F}_t)$ defined by $\mathcal{F}_t = \sigma\{\, B_s : s \leq t \,\}$, will be interpreted as information available up to time t. That is, measurability with respect to $\mathcal{F}_t$ is equivalent to measurability with respect to market information up to time t. Given the structure of transaction costs, consumption and investment decisions are made at intervals. During each interval there is no transaction. All dividends of risky security are re-invested continually in the risky security, and all interest income is re-invested continually in the riskless security.

The investor chooses instants of time at which to process information and make consumption and investment decisions. In other words, information is available continuously through the filtration $\{\mathcal{F}_t : t \geq 0\}$. The investor receives information via controllable filtration

$$\mathbf{H} = \{\mathcal{H}_t : t \geq 0\} \text{ with } \mathcal{H}_t = \mathcal{F}_t,\ t \in [\tau_n, \tau_{n+1}),$$

where τ_n is a $\mathcal{H}_{\tau_{n-1}}$-measurable stopping time at which the n-th transaction occurs. The filtration $\mathbf{H}$ is controllable in the sense that the investor is allowed to choose any sequence $\tau = \{\tau_n : n = 1, 2, 3, ...\}$ of such transaction times with $\tau_1 \equiv 0$. Let $T = \{T_n = \tau_{n+1} - \tau_n : n = 1, 2, 3, ...\}$ denotes the corresponding sequence of transaction intervals. Finding an optimal stopping policy τ is clearly equivalent to finding an optimal transaction interval policy T.

2.5. **The Model.**

2.5.1. *Preferences.* Let the consumption space $\mathcal{C}$ for the investor consists of positive $\mathbf{H}$-adapted consumption processes $\mathbf{C} = \{C_t : t \geq 0\}$ satisfying $\int_0^t C_s ds < \infty$ almost surely for all $t \geq 0$, and

$$E[\int_0^\infty e^{-\delta\, t} u(C_t) dt] < \infty, \tag{2.3}$$

where E denotes the *expected value function*, with respect to P, $T_f < \infty$ is the final time, δ is a strictly positive scalar discount factor and the utility function u, is one of the HARA (hyperbolic absolute risk-aversion) type function, as defined in Merton (1971). We take u as given by

$$u(C) = \frac{1}{\gamma} C^\gamma,\ \ 0 < \gamma < 1. \tag{2.4}$$

2.5.2. *Feasible policies.* Let $\tau = \{\tau_n : n = 1, 2, 3, ...\}$ be sequence of transaction times with $\tau_1 \equiv 0$. Let $T = \{T_n = \tau_{n+1} - \tau_n, n = 1, 2, 3, ...\}$ be the sequence of corresponding transaction intervals. Let $W = \{W_{\tau_n} : n = 1, 2, 3, ...\}$ be the sequence of money withdrawal processes, and $V = \{V_{\tau_n} : n = 1, 2, 3, ...\}$ be the sequence of investment for the risky security.

Let $\mathcal{T}$ denote the space of sequences of strictly positive transaction intervals, $\mathcal{W}$ the space of positive **H**-adapted money withdrawal processes, and $\mathcal{V}$ the space of **H**-adapted investment processes for the risky security. Let $\mathcal{U} = \mathcal{T} \times \mathcal{W} \times \mathcal{V} \times \mathcal{C}$.

Definition 2.6. A budget policy is a quadruplet $(T, W, V, C) \in \mathcal{U}$.

We characterize budget feasible policies as follows. Let $\mathcal{U}$ denotes a class of budget policies. Given a policy $(T, W, V, C) \in \mathcal{U}$, then the money holding at any time t is defined by

$$M_t = \sum_{\{n:\tau_n \le t\}} [\, W_{\tau_n} - \Psi(W_{\tau_n})\,] - \int_0^t C_s \, ds, \tag{2.5}$$

where Ψ is the fixed (bullet) transaction costs function.

Let X_{τ_n} denotes the total wealth invested in the securities at time τ_n, before the nth transaction. Let W_{τ_n} denotes the amount of money withdrawn at time τ_n from the total wealth X_{τ_n}, and V_{τ_n} denotes the market value of the investment in the risky security chosen at time τ_n. After an amount W_{τ_n} is withdrawn from the total wealth X_{τ_n}, and a fraction ε of the remainder, is paid as management fees, then the wealth left for re-investment is $Z_{\tau_n} = (\,1 - \varepsilon)\,[\,X_{\tau_n} - W_{\tau_n}\,]$. Of this amount, V_{τ_n} is invested in the risky security with a per-dollar payback of Γ_{n+1} at the next transaction date, including continually re-invested dividends. And the remainder, $Z_{\tau_n} - V_{\tau_n}$, is invested in the riskless security at the continuously compounding interest rate $r > 0$.

The investor's total wealth invested at the time of the $(n + 1)$th transaction is therefore, for $n = 1, 2, 3, ...,$

$$X_{\tau_{n+1}} = (\,1 - \varepsilon\,)\,[\,X_{\tau_n} - W_{\tau_n}\,]\, e^{r\,T_n} + V_{\tau_n}\,[\,\Gamma_{n+1} - e^{r\,T_n}\,]. \tag{2.6}$$

According to the equation (2.2) and the Itô's formula,[2] the return of the risky investment Γ satisfies

$$\Gamma_{n+1} = \exp\,[(\alpha - \frac{1}{2}\,\sigma^2)\,T_n + \sigma\,(B_{\tau_{n+1}} - B_{\tau_n})]. \tag{2.7}$$

Since $M_0 = 0$, then X_0 is considered as the initial wealth endowment for the investor.

[2]Details may be found in Karatzas and Shreve (1988), or Protter (1990)

Definition 2.7. The budget policy $(T, W, V, C) \in \mathcal{U}$ is budget feasible policy if the associated money process M of (2.5) and invested wealth process X of (2.6) are non-negative.

3. Optimal Control Statement of the Problem

Definition 3.1. Let $\mathcal{U}$ be the set of all budget feasible policies as defined previously. The optimal control problem for the investor is to maximize

$$U(X_0) \equiv \max_{(T,W,V,C)\in\mathcal{U}} E\,[\int_0^\infty e^{-\delta t}\, u(C_t)\, dt], \tag{3.1}$$

subject to, for $n = 1, 2, 3, ...,$

$$X_{\tau_{n+1}} = (\,1 - \varepsilon\,)\,[\,X_{\tau_n} - W_{\tau_n}\,]\, e^{r\,T_n} + V_{\tau_n}\,[\,\Gamma_{n+1} - e^{r\,T_n}\,], \tag{3.2}$$

with $M_t \geq 0$, and $X_{\tau_{n+1}} \geq 0$.

We assume that only money is available to the investor as a medium of exchange and numeraire in the economy. Only money is exchangeable for consumption. It is also assumed that money cannot be borrowed, it can only be acquired by selling the securities, and it is put in the purse M. Because there exists a riskless security with a positive interest rate in the economy, there is no investment demand for money. Duffie and Sun (1990) argued that it will not be optimal for the investor to withdraw more money than the amount needed for financing consumption before the next transaction.

The following result is similar to those in Duffie and Sun (1990).

Theorem 3.2. *Let the value function U be defined as in (3.1), and the transaction costs function $\Psi(W_{\tau_n}) = b$, $b \geq 0$. Then the optimal policy (T, W, V, C) must satisfy for all $n = 1, 2, 3, ...$*

$$\int_{\tau_n}^{\tau_{n+1}} C_t\, dt = W_{\tau_n} - b. \tag{3.3}$$

Proof : Let (T, W, V, C) be an optimal policy. Suppose that there exists an interval $T_j = \tau_{j+1} - \tau_j$ such that $d > 0$ where d is defined by

$$d = W_{\tau_j} - b - \int_{\tau_j}^{\tau_{j+1}} C_t\, dt.$$

Because there exists a riskless security with a positive interest rate, then the investor will be better off if the amount d is invested in the riskless security during the interval T_j, and the interest income $d\,(e^{r\,T_j} - 1)$ is consumed in the next interval. In other words, the optimal policy (T, W, V, C) is dominated by a feasible policy $(T, \overline{W}, V, \overline{C})$, which is defined by

$$\overline{W}_{\tau_j} = b + \int_{\tau_j}^{\tau_{j+1}} C_t\, dt,$$

$$\overline{W}_{\tau_{j+1}} = W_{\tau_{j+1}} + d\, e^{rT_j} > W_{\tau_{j+1}},$$

$$\overline{C}_t = C_t + \frac{1}{T_{j+1}}\, d\,(e^{rT_j} - 1) > C_t, \quad t \in [\tau_{j+1}, \tau_{j+2}),$$

$$\overline{C}_t = C_t, \quad \overline{W}_{\tau_j} = W_{\tau_j}, \quad \text{otherwise.}$$

This contradicts with the fact that $(T, W, V, C,)$ is optimal. Therefore, for all n,

$$\int_{\tau_n}^{\tau_{n+1}} C_t\, dt \geq W_{\tau_n} - b.$$

On the other hand, all of expenditures must be financed from the stock of money. Therefore, for all n,

$$\sum_{i}^{n} \int_{\tau_i}^{\tau_{i+1}} C_t\, dt \leq \sum_{i}^{n} [\, W_{\tau_i} - b\,].$$

Therefore, for all n, an optimal policy (T, W, V, C) must satisfy

$$\int_{\tau_n}^{\tau_{n+1}} C_t\, dt = W_{\tau_n} - b.$$

Hence, the proof of Theorem 3.2 has been completed ■

Corollary 1. By the definition of money holding M_t of equation (2.5), then

$$M_{\tau_n} = W_{\tau_n} - b, \quad n = 1, 2, 3, ...$$

Therefore, the optimal control problem (3.1)-(3.2) is equivalent to the optimal control problem :

$$U(X_0) = \max_{(T,W,V,C)\in\mathcal{U}} E\, [\int_0^\infty e^{-\delta\, t}\, u(C_t)\, dt\,] \tag{3.4}$$

subject to, for $n = 1, 2, 3, ...,$

$$\int_{\tau_n}^{\tau_{n+1}} C_t\, dt = W_{\tau_n} - b, \tag{3.5}$$

$$X_{\tau_{n+1}} = (1 - \varepsilon)\, [\, X_{\tau_n} - W_{\tau_n}\,]\, e^{r\, T_n} + V_{\tau_n}\, [\, \Gamma_{n+1} - e^{r\, T_n}\,] \geq 0. \tag{3.6}$$

The term under the expectation in (3.4) may be re-written as :

$$\int_0^\infty e^{-\delta\, t}\, u(C_t)\, dt = \sum_{n=1}^{\infty} e^{-\delta\, \tau_n} \int_{\tau_n}^{\tau_{n+1}} e^{-\delta\, (t - \tau_n)}\, u(C_t)\, dt.$$

Therefore, the equation (3.4) may be re-written as :

$$U(X_0) = E\, [\sum_{n=1}^{\infty} e^{-\delta\, \tau_n} \int_{\tau_n}^{\tau_{n+1}} e^{-\delta\, (t - \tau_n)}\, u(C_t)\, dt\,].$$

Hence, the control problem (3.4) - (3.6) can be solved in two steps.

In the first step, the control problem for consumption between transaction intervals is solved for any given budget feasible (T, W, V). This control problem is a deterministic continuous-time control problem, because the consumption C is adapted to the filtration H. Let the objective function for this problem be denoted by J. In the second step, the investor chooses a budget feasible (T, W, V) to maximize $E\,[\,\sum_{n=1}^{\infty}\; e^{-\delta\;\tau_n}\; J(T_n, M_{\tau_n})\,]$. This is similar to a stochastic discrete-time control problem except that the sequence T of transaction intervals is controllable.

Consider an investor with an initial money endowment Z and time-horizon t. The deterministic control problem for the investor is to maximize the objective function

$$J(t, Z) \equiv \int_0^t \exp(-\delta s)\; u(C_s)\; ds \tag{3.7}$$

over $\{C_s : 0 \le s \le t\}$, subject to :

$$\int_0^t C_s\;\; ds \quad \le \quad Z. \tag{3.8}$$

Theorem 3.3. *The optimal value function J for the deterministic control problem (3.7) - (3.8) satisfies*

$$J(t, Z) = (\frac{1-\gamma}{\delta})^{1-\gamma}\; [1 - \exp(-\frac{\delta}{1-\gamma}\; t)]^{1-\gamma}\; \frac{1}{\gamma}\; Z^{\gamma}. \tag{3.9}$$

Proof : The above problem falls in the category of isoparametric problem in the calculus of variations (see Alekseev and others(1987)). Hence, the above problem can be solved by a Lagrange multiplier technique. Since the problem is to maximize consumption, then the consumption can always be increased such that the left hand side of equation (3.8) is equal to the right hand side of equation (3.8). By appending (3.8) into (3.7), then a Lagrange multiplier gives

$$L = \int_0^t [\exp(-\delta s)\; u(C_s) - \lambda\, C_s]\; ds + \lambda\, Z,$$

where λ is a Lagrange multiplier. A necessary condition for C to maximize the augmented integrand of L is that it satisfies the Euler equation

$$\exp(-\delta\tau)\; u'(C_\tau) \;=\; \lambda.$$

Therefore, the Lemma is proved by solving the following problem:

$$\exp(-\delta\tau)\; u'(C_\tau) \;=\; \lambda, \tag{3.10}$$

where u' denotes the first derivative of the utility function u , C_τ denotes the optimal consumption at time τ.

From the definition of the utility function u in equation (2.4), then its first derivative u' is given by

$$u'(C_\tau) \;=\; C_\tau^{\gamma-1}, \qquad 0<\gamma<1.$$

Then by substituting this derivative into (3.10) yields

$$C_\tau = [\ \lambda\ \exp(\delta\,\tau)\ ]^{-1/(1-\gamma)}. \tag{3.11}$$

Since the control problem is to maximize the utility function, then C_τ optimal must satisfy

$$Z \;=\; \int_0^t C_\tau\ \ d\tau.$$

Use the last equation and equation (3.11) to get

$$C_\tau = (\frac{1-\gamma}{\delta})^{-1}\ [1-\exp(-\frac{\delta}{1-\gamma}\,t)]^{-1}\ Z\ \exp(-\frac{\delta}{1-\gamma}\,\tau). \tag{3.12}$$

Finally, by insertion of equation (3.12) into (3.7) results in

$$J(t,Z) = (\frac{1-\gamma}{\delta})^{1-\gamma}\ [1-\exp(-\frac{\delta}{1-\gamma}\,t)]^{1-\gamma}\ \frac{1}{\gamma}\,Z^\gamma \quad \blacksquare$$

Since $0<\gamma<1$, then without loss of the generality, the term $(\frac{1-\gamma}{\delta})^{1-\gamma}$ will be left out in future discussions. Therefore, by Corollary 1,

$$J(T_n, M_{\tau_n}) = [1-\exp(-\frac{\delta}{1-\gamma}\,T_n)]^{1-\gamma}\ \frac{1}{\gamma}\,(W_{\tau_n}-b)^\gamma.$$

Now let $Q_n = 1-\exp(-\frac{\delta}{\nu}\,T_n)$, with $\nu = 1-\gamma$. Then we have the modified stochastic optimal control problem as given by

$$U(X_0) = \max_{\{T\in\mathcal{T}, W\in\mathcal{W}, V\in\mathcal{V}\}} E\,[\sum_{n=1}^{\infty} e^{-\delta\tau_n}\,Q_n^\nu\,\frac{1}{\gamma}\,(\,W_{\tau_n}-b\,)^\gamma], \tag{3.13}$$

subject to

$$X_{\tau_{n+1}} = (1-\varepsilon)\ [\ X_{\tau_n}-W_{\tau_n}\ ]\ e^{r\,T_n} + V_{\tau_n}\ [\ \Gamma_{n+1}-e^{r\,T_n}\ ] \geq 0, \tag{3.14}$$

where $T_n = \tau_{n+1}-\tau_n$ for $n=1,2,3,....$

The application of Bellman principle on U, results in

$$U(X_{\tau_n}) = \max_{\{T_n, W_{\tau_n}, V_{\tau_n}\}} \{Q_n^\nu\,\frac{1}{\gamma}\,(W_{\tau_n}-b)^\gamma + e^{-\delta T_n}\,E\,[U(X_{\tau_{n+1}})\mid \mathcal{H}_{\tau_n}]\}, \tag{3.15}$$

subject to

$$X_{\tau_{n+1}} = (1-\varepsilon)\ [\ X_{\tau_n}-W_{\tau_n}\ ]\ e^{r\,T_n} + V_{\tau_n}\ [\ \Gamma_{n+1}-e^{r\,T_n}\ ], \tag{3.16}$$

for $n=1,2,3,....$

We summarize the problem faced by the investor in the following definition.

Definition 3.4. Let $\mathcal{U}$ be the set of all budget feasible policies as defined previously. The optimal control problem for the investor is to maximize

$$U(X_{\tau_n}) = \max_{\{T_n, W_{\tau_n}, V_{\tau_n}\}} \{Q_n^{\nu} \frac{1}{\gamma} (W_{\tau_n} - b)^{\gamma} + e^{-\delta T_n} E\left[U(X_{\tau_{n+1}}) \mid \mathcal{H}_{\tau_n}\right]\}, \tag{3.17}$$

subject to

$$X_{\tau_{n+1}} = (1-\varepsilon)\,[\,X_{\tau_n} - W_{\tau_n}\,]\,e^{r\,T_n} + V_{\tau_n}\,[\,\Gamma_{n+1} - e^{r\,T_n}\,], \tag{3.18}$$

for $n = 1, 2, 3, ...,$ with $M_t \geq 0,$ and $X_{\tau_{n+1}} \geq 0.$

The following result is proved in Duffie and Sun (1990) but is given for completeness.

Lemma 3.5. *Let* $\overline{Q}(n) = [1 - \exp(-\frac{\delta}{\nu}\,T_n)]^{\nu}$. *Suppose that* f *is a real-valued function on* $[\,0,\ \infty)$ *satisfying the two conditions :*
(i) For all $n = 1, 2, 3, ...,$

$$f(X_{\tau_n}) = \max_{(T_n, W_{\tau_n}, V_{\tau_n})} \{\,\overline{Q}(n)\,\frac{1}{\gamma}\,(W_{\tau_n} - b)^{\gamma} + E\,[\,e^{-\delta\,T_n}\,f(X_{\tau_{n+1}})\,]\mid \mathcal{H}_{\tau_n}\,\}. \tag{3.19}$$

(ii) For any feasible policy,

$$\lim_{n\,\to\,\infty} E\,[\,e^{-\delta\,\tau_n}\,f(X_{\tau_n})\,] = 0. \tag{3.20}$$

If (T^*, W^*, V^*) *achieves the maximum in (3.19) for all n then* f *is the value function for the control problem (3.13), and* (T^*, W^*, V^*) *is an optimal policy.*

Proof : Let $n = 1,$ to begin with, that is $\tau_1 = 0.$ Then

$$\begin{aligned} f(X_0) &= \max_{(T_1, W_0, V_0)} \{\,\overline{Q}(1)\,\frac{1}{\gamma}\,(W_0 - b)^{\gamma} + E\,[e^{-\delta\,T_1} f(X_{\tau_2})]\} \\ &\geq \overline{Q}(1)\,\frac{1}{\gamma}\,(W_0 - b)^{\gamma} + e^{-\delta\,T_1}\,E\,[\,f(X_{\tau_2})\,] \end{aligned}$$

for any feasible $T_1,\ W_0,\ V_0.$ By induction, for any $(T, W, V) \in \mathbf{T} \times \mathbf{W} \times \mathbf{V}$ then

$$f(X_0) \geq E\,[\,\sum_{i=1}^{n}\,e^{-\delta \tau_i}\,\overline{Q}(i)\,\frac{1}{\gamma}\,(W_{\tau_i} - b)^{\gamma} + e^{-\delta \tau_{n+1}}\,f(X_{\tau_{n+1}})\,].$$

Let $n\ \to\ \infty,$ it follows by condition (ii) of Lemma 3.5 that

$$f(X_0) \geq E\,[\,\sum_{n=1}^{\infty}\,e^{-\delta\,\tau_n}\,\overline{Q}(n)\,\frac{1}{\gamma}\,(W_{\tau_n} - b)^{\gamma}\,].$$

This holds for an arbitrary feasible policy (T, V, W) . Hence,

$$f(X_0) \geq U(X_0).$$

On the other hand, $U(X_0) \geq f(X_0)$ by the definition of $U(X_0)$. Henceforth, $f(X_0) = U(X_0)$, and consequently (T^*, W^*, V^*) is optimal ■

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